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On the Character Table of the Multiplicative Group of a Finite Nearfield

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Abstract This study investigates the representation theory of the multiplicative group of a finite nearfield, with a specific focus on the group $G_{q,n}$, arising from Dickson nearfields. Nearfields, generalizations of fields lacking one of the two distributive laws, pose interesting challenges in algebraic structures due to their noncommutativity and asymmetry. Their multiplicative groups, particularly those derived from Dickson nearfields, have a metacyclic structure that allows for intricate analysis using representation theory. The purpose of this research is to construct and analyze the character tables of these multiplicative groups. We explore foundational concepts such as Schur's lemma, Maschke's theorem, and orthogonality relations, and apply them within the context of finite nearfields. A combination of theoretical and computational methodologies is employed. Specifically, we make use of the Magma algebra system to build and verify character tables for $G_{q,n}$ for small values of q and n, including detailed derivations for $G_{3,2}$ and $G_{4,3}$. The principal results include explicit character tables, classifications of conjugacy classes, degrees of irreducible representations, and a generalization framework for any pair (q, n). The research demonstrates that the multiplicative groups $G_{q,n}$ are not only metacyclic but also nilpotent and solvable under specific conditions. It further establishes structural conditions that determine the conjugacy class partitioning and representation degrees. Our conclusions highlight the deep connection between nearfield algebraic properties and finite group representation theory. This work contributes to the theoretical development of group representations in non-classical algebraic settings and opens doors to applications in geometry, coding theory, and cryptography. Practically, the use of Magma facilitates automated representation analysis, and socially, this research promotes a broader understanding of non-traditional algebraic systems. While the current study is computationally verified for small instances, future work will aim to optimize algorithms for handling larger $G_{q,n}$ structures, enhancing scalability and applicability.

Keywords Nearfield, Character Table, Group $G_{q,n}$

1 Introduction

Nearfields were first studied by Dickson in 1905 and were applied immediately by mathematicians to geometry. Lacking one side distributive law makes the study of nearfields difficult despite they look a lot like fields. Zassenhauss [1], Dancs [2, 3], Karzel and Ellers [4] have solved some important problems in this area. Recently the author in [5] has investigated on the generalized distributive set of a finite nearfield.

The concept of representations forms a core area of mathematics, with wide-ranging applications in algebra, geometry, and physics [6]. Simply put, it studies how algebraic structures such as groups can be represented by matrices and how these matrices can be used to understand the properties of these groups. The study of representations of limited groups, in particular, is a fruitful and well-developed subject, offering powerful tools for the analysis of symmetries and complex algebraic structures.

Near-vector spaces were initially introduced by André [7] in 1974. It represents an extension of traditional vector spaces. Other types of near-vector spaces have been studied

in [8, 9, 10]. These structures are defined over near-fields, which themselves generalize fields. In this study, the nearfield will lead us to examine the Multiplicative Group denoted $G_{q,n}$, which is a finite nearfield and the central element of this research. In this paper, we focus on the representation theory of the multiplicative group of finite nearfield, exploring how concepts and techniques from group representation theory can be extended and utilized for these more general structures. Our objective is to develop a profound comprehension of the attributes and properties of multiplicative group of finite nearfield through the prism of representation theory.

This work is structured around three sections. Section 2 discusses near-fields, defining and exploring their properties, with a particular emphasis on finite near-fields and the multiplicative group of the Dickson near-field and their characteristics. In Section 3, the analysis extends to characters and properties of orthogonality, accompanied by examples of character tables for certain finite groups and multiplicative groups of the Dickson near-field (denoted $G_{q,n}$). The section also generalizes these concepts for any pair (q, n) and demonstrates the use of software like Magma for constructing character tables.

2 Some remarks on nearfields

Let S be any group with identity 0. We will use S^* to denote $S \setminus \{0\}$.

Definition 1. ([11]) Let $(R, +, \cdot)$ be a triple such that $(R, +)$ is a group, $(R, \cdot)$ is a semigroup, and $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$. Then $(R, +, \cdot)$ is a (left) nearring. If in addition $\left(R^*, \cdot\right)$ is a group then $(R, +, \cdot)$ is called a nearfield.

So a nearfield is an algebraic structure similar to a skewfield (sometimes called a division ring) except that it has only one of the two distributive laws. It is well known that the additive group of a (left) nearfield is abelian, see for instance [1, 14]. Throughout this paper we will make use of (left) nearfields.

Furthermore, as we know from the definition of a (left) nearfield, we do not necessarily have the right distributivity law and commutativity of multiplication. For this reason, the following concepts have been defined and they will be used in the next chapters.

Definition 2. ([12]) Let R be a nearfield.

- The multiplicative center of $(R, \cdot)$ denoted by $C(R)$, is defined as follows:

$$C(R) = \{x \in R : x \cdot y = y \cdot x \text{ for all } y \in R\}.$$

  i.e., it is the set of elements of R that commute with every element of R.

- We use $D(R)$ to denote the set of all distributive elements of R, also called the kernel of $(R, +, \cdot)$. It is defined as follows:
  $D(R) = \{\lambda \in R : \ (\alpha+\beta)\cdot\lambda = \alpha\cdot\lambda + \beta\cdot\lambda \ \forall \ \alpha, \beta \in R\}$.

2.1 Finite Dickson nearfield

To construct finite Dickson nearfields, we need two concepts: Dickson pair and coupling map.

Definition 3. ([12]) A pair of positive integers (q, n) is called a Dickson pair if the following conditions are satisfied:

(i) q is some power p^l of some prime p,

(ii) each prime divisor of n divides $q - 1$,

(iii) $q \equiv 3 \bmod 4$ implies 4 does not divide n.

Let (q, n) be a Dickson pair and $k \in \{1, \ldots, n\}$. We will denote the positive integer $\frac{q^k-1}{q-1}$ by $[k]_q$.

Definition 4. ([12]) Let R be a nearfield and $Aut(R, +, \cdot)$ the set of all automorphisms of R. A map $\phi : \ R^* \to Aut(R, +, \cdot)$ defined by $a \mapsto \phi_a$ is called a coupling map if for all $a, b \in R^*$, $\phi_a \circ \phi_b = \phi_{\phi_a(b)\cdot a}$.

Lemma 5. *([12, 13]) Let (q, n) be a Dickson pair. Then $\left\{[k]_q : 1 \leq k \leq n\right\}$ forms a finite complete set of different residues modulo n.*

Every Dickson pair (q, n) gives rise to a finite Dickson nearfield. This is obtained by replacing the usual multiplication $\cdot$ in the finite field $\mathbb{F}_{q^n}$ of order q^n with a new multiplication $\circ$. We shall denote the set of Dickson nearfields arising from the Dickson pair (q, n) by $DN(q, n)$ and the Dickson nearfield arising from the Dickson pair (q, n) with generator g by $DN_g(q, n)$. Furthermore in [12], the new multiplication is constructed as follows:

Let g be such that $\mathbb{F}_{q^n}^* = \langle g \rangle$ and $H = \langle g^n \rangle$. The quotient group is given by

$$\begin{aligned}\mathbb{F}_{q^n}^*/H &= \left\{g^{[1]_q}H, g^{[2]_q}H, \ldots, g^{[n]_q}H\right\} \\ &= \left\{H, gH, \ldots, g^{n-1}H\right\}.\end{aligned}$$

The coupling map ϕ is defined as

$$\begin{array}{ccc}\mathbb{F}_{q^n}^* & \to & Aut(\mathbb{F}_{q^n}, +, \cdot) \\ \alpha & \mapsto & \phi_\alpha = \varphi^k(\alpha)\end{array}$$

where φ is the Frobenius automorphism of $\mathbb{F}_{q^n}$ and k is a positive integer ($k \in \{1, \ldots, n\}$) such that $\alpha \in g^{[k]_q}H$. Let $\alpha, \beta \in \mathbb{F}_{q^n}$, then we have

$$\begin{aligned}\alpha \circ \beta &= \begin{cases} \alpha \cdot \phi_\alpha(\beta) & \text{if} \quad \alpha \neq 0 \\ 0 & \text{if} \quad \alpha = 0 \end{cases} \\ &= \begin{cases} \alpha \cdot \varphi^k(\beta) & \text{if } \alpha \in g^{[k]_q}H \\ 0 & \text{if } \alpha = 0 \end{cases} \\ &= \begin{cases} \alpha \cdot \beta^{q^k} & \text{if } \alpha \in g^{[k]_q}H \\ 0 & \text{if } \alpha = 0 \end{cases}\end{aligned}$$

for $k \in \{1, \ldots, n\}$. Thus $DN_g(q, n) := \left(\mathbb{F}_{q^n}, +, \circ\right)$ is the finite Dickson nearfield constructed by taking $H = \langle g^n \rangle$. By taking all Dickson pairs, all finite Dickson nearfields arise in this way [14]. Furthermore we deduce the following.

Lemma 6. *([13]) Let (q,n) be a Dickson pair with $q=p^l$ for some prime p and positive integers l,n. Let g be a generator of $\mathbb{F}_{q^n}^*$ and R the finite nearfield constructed with $H=\langle g^n\rangle$. Then n divides $[n]_q$ and $g^{[n]_q}H=H$.*

Thirty years after Dickson's work, Zassenhauss fundamentally determined all finite nearfields.

Theorem 7. *([13]) A finite nearfield is either a finite Dickson nearfield or it is one of the 7 exceptional nearfields of order $5^2, 7^2, 11^2, 23^2, 29^2, 59^2$.*

Note that there exist two exceptional nearfields of order 11^2 (see [13] for more details and Table 1.). He described these seven exceptional nearfields by denoting them as $N_1, N_2, N_3, N_4, N_5, N_6, N_7$ with orders 5^2, 7^2, 11^2, 23^2, 29^2, 59^2, respectively.

Table 1. The 7 exceptional nearfields [13].

i	p	Order of N_i	$(N_i^\times,\cdot)$ is generated by
1	5	5^2	$\begin{pmatrix}1&-2\\-1&-2\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
2	11	11^2	$\begin{pmatrix}1&5\\-5&-2\end{pmatrix}$ and $\begin{pmatrix}4&0\\0&4\end{pmatrix}$
3	7	7^2	$\begin{pmatrix}1&3\\-1&-2\end{pmatrix}$
4	23	23^2	$\begin{pmatrix}1&-6\\12&-2\end{pmatrix}$ and $\begin{pmatrix}2&0\\0&2\end{pmatrix}$
5	11	11^2	$\begin{pmatrix}2&4\\1&-3\end{pmatrix}$
6	29	29^2	$\begin{pmatrix}1&-7\\-12&-2\end{pmatrix}$ and $\begin{pmatrix}-13&0\\0&-13\end{pmatrix}$
7	59	59^2	$\begin{pmatrix}9&15\\-10&-10\end{pmatrix}$ and $\begin{pmatrix}4&0\\0&4\end{pmatrix}$

Building on the previous lemmas, we categorize the structure of nearfields based on their order and isomorphisms to specific groups.

Theorem 8. *([13]) Every such structure N falls into one of two categories: it is either a Dickson nearfield, or its order $|N|$ equals p^2 for a specific prime p, under the following scenarios:*

1. *When $p=5$, the multiplicative group N^* is isomorphic to the special linear group $SL(2,3)$.*
2. *For $p=7$, N^* is isomorphic to the double cover of the symmetric group $2S_4$.*
3. *If $p=11$, N^* can be isomorphic to either $SL(2,3)\times C_5$ or $SL(2,5)$.*
4. *In the case of $p=23$, N^* is isomorphic to $2S_4\times C_{11}$.*
5. *For $p=29$, N^* is isomorphic to $SL(2,5)\times C_7$.*
6. *When $p=59$, N^* is isomorphic to $SL(2,5)\times C_{29}$.*

In each of these seven unique cases, there is precisely one distinct isomorphism type for N.

Finally, we categorize finite nearfields into three distinct classes based on the findings of [16]:

1. N is a finite field.
2. N is a Dickson nearfield.
3. N is one of the seven exceptional nearfields of orders $5^2, 7^2, 11^2, 23^2, 29^2, 59^2$.

Remark 9. ([12]) All finite nearfields up to the seven exceptional cases are finite Dickson nearfields i.e., finite Dickson nearfields that arise from the pair (q,n).

2.2 Some remarks on the multiplicative group of a finite Dickson nearfield

In this section we will see a particular difference between finite field and finite nearfield. We will consider automorphisms of the multiplicative group of nearfield in order to construct near vector space that we will see later on.

Definition 10. ([13]) A metacyclic group is a group G such that both its commutator subgroup $G'=[G,G]$ and the quotient group G/G' are cyclic.

Assume that (q,n) is a Dickson pair. We have seen in the subsection 2.4 that for all (q,n) we can construct an associated Dickson nearfield $DN(q,n)$. Let $G_{q,n}$ denote the multiplicative group of $DN(q,n)$. It is generated by two elements and can be presented as follows.

Lemma 11. *[13] The multiplicative group of a finite Dickson nearfield $DN(q,n)$ can be presented as*

$$G_{q,n}=\langle a,b : a^m=1, b^n=a^t, ba=a^qb\rangle$$

where $m=\frac{q^n-1}{n}$ and $t=\frac{m}{q-1}$.

Remark 12. ([16]) The multiplicative group of $DN(q,n)$ is metacyclic if every element of the group is of the form $a^\alpha b^\beta$ where

$$\left(a^kb\right)^\beta=a^{k(q^{\beta-1}+\cdots+q+1)}b^\beta=a^{k\frac{1-q^\beta}{1-q}}b^\beta$$

such that a,b are the generators of the group.

Given a nearfield N, a way to check if it is Dickson nearfield is to find a field F such that N is its $\phi-$derivation. Alternatively, we can use the following axiom.

Theorem 13. *[14] A finite nearfield N is a Dickson nearfield if and only if $G:=(N^\times,\cdot)$ is metacyclic.*

Example 14. All finite fields are Dickson nearfields because their multiplicative groups are cyclic. Hence their multiplicative groups are metacylic.

For a given Dickson pair (q,n), the corresponding Dickson nearfield is represented as $DF(q,n)$, with its multiplicative group designated as $G_{q,n}$. The group $G_{q,n}$ is metacyclic and can be presented as

$$G_{q,n}=\langle a,b \mid a^m=1, b^n=a^t, ba=a^qb\rangle,$$

where $m = \frac{q^n-1}{n}$ and $t = \frac{m}{q-1}$.

The order of this group is an essential property to note: $|G_{q,n}| = q^n - 1$. Next, we present a theorem that details the properties and subgroups of the multiplicative group $G_{q,n}$.

Theorem 15. *([16]) Let (q, n) be a Dickson pair, $m = \frac{q^n-1}{n}$, and $t = \frac{m}{q-1}$.*

(o) $Z(G_{q,n}) = \langle a^t \rangle$.

(o) The Sylow subgroups of $G_{q,n}$ are cyclic or quaternion.

(o) $\gcd(n, t) = \gcd(q-1, t) \leq 2$ *and the following are equivalent:*

(I) The 2-Sylow subgroups are cyclic.

(II) $\gcd(q - 1, t) = 1$.

(III) n *is odd or* $q \equiv 3 \pmod 4$.

(o) $\langle a \rangle$ is a maximal abelian subgroup of $G_{q,n}$.

(o) If $(q, n) \neq (3, 2)$, then $\langle a \rangle$ contains all abelian normal subgroups of $G_{q,n}$. In particular, $\langle a \rangle$ is characteristic in $G_{q,n}$.

2.2.1 Metacyclic Group

Three different types of metacyclic groups emerge from its structure which are: split metacyclic groups, metacyclic p-groups, and metacyclic $\{p, q\}$-groups. Let us delve into each type. We are now in a position to prove the following result.

Theorem 16. *Consider a Dickson pair (q, n) characterized by $q = p^l$, where p is a prime and l and n are positive integers. Its multiplicative group, denoted $G_{q,n}$, is a metacyclic p-group.*

Proof. We proceed by elimination:

1. Split Metacyclic Group: The key relation $ba = a^q b$ shows how conjugation by b affects a. If $G_{q,n}$ were split metacyclic, there would exist an element b' such that $b'a = ab'$ and b' has order n. This would mean that b and a commute, which contradicts the given relation $ba = a^q b$ unless $q = 1$, which cannot be the case because q is a power of a prime p and p is greater than 1. Thus $G_{q,n}$ is not Split metacylic group.

2. Metacyclic p-Group: For $G_{q,n}$ to be a metacyclic p-group, the order of the group must be a power of a prime p. Since q is a power of a prime p, we would need n to also be a power of p for the group order to be a power of p. The order of the group is related to the order of the elements a and b, which come from the relations $a^m = 1$ and $b^n = a^t$. If both m and n are powers of the prime p, then $G_{q,n}$ would be a metacyclic p-group.

3. Metacyclic $\{p, q\}$-Group: This type of group would involve the orders of the cyclic subgroups being powers of distinct primes. However, in the context of a Dickson nearfield, this categorization doesn't quite fit because by the definition of a Dickson pair, n relates to $q - 1$, and q is a power of a prime p. Therefore, any prime that divides n also divides $q - 1$, so the orders of the cyclic subgroups won't be powers of distinct primes but rather just powers of the prime p. This suggests that the notion of a metacyclic $\{p, q\}$-group doesn't apply in the strict sense to Dickson nearfields as defined by the Dickson pair criteria.

Therefore, $G_{q,n}$ is a metacyclic p-group. □

2.2.2 Solvable and Nilpotent group

Let (q, n) be a Dickson pair such that q equals p^l, where p is a prime number and l and n are positive integers. It is clear that the multiplicative group corresponding represented as $G_{q,n}$ is solvable. Now, we can state that.

Theorem 17. *Let (q, n) denote a Dickson pair with q being p^l, where p is a prime number and l and n are positive integers. Its multiplicative group noted $G_{q,n}$ is nilpotent.*

Proof. The commutator subgroup, denoted as $[G_{q,n}, G_{q,n}]$, is the subgroup of $G_{q,n}$ generated by all commutators of the form $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in G_{q,n}$. The commutators are:

$$[a, b] = b^{-1}a^{-1}ab = b^{-1}a^{-1+q}b = b^{-1}a^t b = a^t,$$

$$[[G_{q,n}, G_{q,n}], [G_{q,n}, G_{q,n}]] = [\langle a^t \rangle, \langle a^t \rangle] = 1.$$

The series of commutator subgroups terminates at the identity element:

$$G_{q,n} \supset [G_{q,n}, G_{q,n}] \supset [[G_{q,n}, G_{q,n}], [G_{q,n}, G_{q,n}]] = 1$$

Therefore, $G_{q,n}$ is nilpotent. □

3 Character Table of $G_{q,n}$

We examine the multiplicative group $G_{q,n}$ in detail, exploring some of its specific characteristics. Additionally, we will use the Magma software to generate these character tables and generalize the conjugacy classes and irreducible representations.

3.1 Character table of $G_{3,2}$

We know that $q = 3$ and $n = 2$. After calculation : $m = 4$ and $t = 2$. So, the group $G_{3,2}$ can be presented as:

$$G_{3,2} = \left\langle a, b \mid a^4 = 1, b^2 = a^2, ba = a^3 b \right\rangle.$$

3.1.1 Conjugacy Classes

The elements of the group are $1, a, a^2, a^3, b, ab, a^2b, a^3b$.

1. For the identity element 1, the conjugacy class is simply $\{1\}$ because $x^{-1}1x = 1$ for any x in the group.

2. For the element a, the conjugacy class is $\{a, a^3\}$. This is because $x^{-1}ax = a$ and $x^{-1}ax = a^3$ for any x in the group.
3. For the element a^2, the conjugacy class is $\{a^2\}$. This is because $x^{-1}ax = a^2$ for any x in the group.
4. For the element b, the conjugacy class is $\{b, a^2b\}$. This is because $x^{-1}bx = b$ and $x^{-1}a^2bx = a^2b$ for any x in the group.
5. For the element ab, the conjugacy class is $\{ab, a^3b\}$. This is because $x^{-1}abx = ab$ and $x^{-1}abx = a^3b$ for any x in the group.

Hence, the conjugacy classes of $G_{3,2}$ are: $\{1\}, \{a^2\}, \{a, a^3\}, \{b, a^2b\}, \{ab, a^3b\}$.

3.1.2 Degree 1 of irreducible representations

In this case, the degree 1 representations are simply homomorphisms from $G_{3,2}$ to $\mathbb{C}^\times$. Since $G_{3,2}$ is a group generated by a and b with the relations $a^4 = 1$, $b^2 = a^2$, and $ba = a^3b$, any homomorphism ρ is completely determined by $\rho(a)$ and $\rho(b)$. Let's find all representations of degree 1:

1. Since $a^4 = 1$, $\rho(a)$ must be a fourth root of unity in $\mathbb{C}$. Therefore, $\rho(a)$ can be 1, -1, i, or $-i$.
2. Similarly, since $b^2 = a^2$, $\rho(b)^2$ must equal $\rho(a)^2$. Given that $\rho(a)$ is a fourth root of unity, $\rho(a)^2$ is a second root of unity. Therefore, $\rho(b)$ must be a complex number whose square is either 1 or -1, which means $\rho(b)$ can be 1, -1, i, or $-i$.
3. The relation $ba = a^3b$ imposes that $\rho(b)\rho(a) = \rho(a)^3\rho(b)$. Since $\rho(a)$ is a fourth root of unity, $\rho(a)^3$ is just the complex conjugate of $\rho(a)$. This means that $\rho(b)$ has to commute with $\rho(a)$, which is true for any choice of $\rho(a)$ and $\rho(b)$ from the above options because complex numbers always commute under multiplication.

With these constraints, we can enumerate all possible degree 1 representations:

- ρ_1: $\rho_1(a) = 1, \rho_1(b) = 1$,
- ρ_2: $\rho_2(a) = 1, \rho_2(b) = -1$,
- ρ_3: $\rho_3(a) = -1, \rho_3(b) = 1$,
- ρ_4: $\rho_4(a) = -1, \rho_4(b) = -1$.

3.1.3 Degree 1 of character representations

- $\chi_1(1) = 1$, $\chi_1(a) = 1$, $\chi_1(a^2) = 1$, $\chi_1(b) = 1$, $\chi_1(ab) = 1$.
- $\chi_2(1) = 1$, $\chi_2(a) = 1$, $\chi_2(a^2) = 1$, $\chi_2(b) = -1$, $\chi_2(ab) = -1$.
- $\chi_3(1) = 1$, $\chi_3(a) = -1$, $\chi_3(a^2) = 1$, $\chi_3(b) = 1$, $\chi_3(ab) = -1$.
- $\chi_4(1) = 1$, $\chi_4(a) = -1$, $\chi_4(a^2) = 1$, $\chi_4(b) = -1$, $\chi_4(ab) = 1$.

3.1.4 Degree 2 of irreducible representations

$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\rho(a)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\rho(ab) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

3.1.5 Degree 2 of character representation

$\chi_5(1) = 2$, $\chi_5(a) = 0$, $\chi_5(a^2) = -2$, $\chi_5(b) = 0$, $\chi_5(ab) = 0$.

Therefore, the character table of $G_{3,2}$ is presented in the Table 2:

Class	1	a^2	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 2. Character table of $G_{3,2}$

3.1.6 Building character table of $G_{3,2}$ in Magma

We use the software Magma to determine the character table of $G_{3,2}$.

Magma code:

```
q := 3; n := 2;
m := (q^n - 1) div (q - 1);
t := m div (q - 1);

F<a, b> := FreeGroup(2);
G := quo< F | a^m = 1, b^n = a^t, b*a = a^q*b >;

// Convert to a permutation group
G_3_2 := PermutationGroup(G);

// Compute the character table
CT := CharacterTable(G_3_2);

// Print the character table
CT;
```

Listing 1. Magma code for $G_{3,2}$

We obtain the following output:

```
Character Table of Group G_3_2
------------------------------


-----------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  4  4  4
-----------------------
p  =  2   1  1  2  2  2
-----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
```

```
X.3   +   1   1   1 -1 -1
X.4   +   1   1 -1 -1   1
X.5   -   2 -2   0   0   0
```

Remark 18. The character table of $G_{3,2}$ is similar to the character tables of Q_8 and D_8.

3.2 Character table of $G_{4,3}$

After calculation: $m = 21$ and $t = 7$. So, the group $G_{4,3}$ can be presented as:

$$G_{4,3} = \langle a, b \mid a^{21} = 1, b^3 = a^7, ba = a^4b \rangle.$$

3.2.1 Conjugacy Classes

The elements of the group are:
$1, a, a^2, \ldots, a^{20}, b, ab, a^2b, \ldots, a^{20}b, b^2, ab^2, \ldots, a^{20}b^2$.
Calculate the conjugacy classes for each element g by using the formula $\{x^{-1}gx \mid x \in G_{4,3}\}$.

1. For the identity element 1, the conjugacy class is simply $\{1\}$ because $x^{-1}1x = 1$ for any x in the group.

2. Elements conjugate of a:

   $ba = a^4b \implies bab^{-1} = a^4$; $ba^4 = a^{16}b \implies ba^4b^{-1} = a^{16}$; $ba^{16} = ab \implies ba^{16}b^{-1} = a$. The conjugate class corresponding is $\{a, a^4, a^{16}\}$.

3. Elements conjugate of a^2:

   $ba^2 = a^8b \implies ba^2b^{-1} = a^8$; $ba^8 = a^{11}b \implies ba^8b^{-1} = a^{11}$; $ba^{11} = a^2b \implies ba^{11}b^{-1} = a^2$. The conjugate class corresponding is $\{a^2, a^8, a^{11}\}$.

4. Elements conjugate of a^3:

   $ba^3 = a^{12}b \implies ba^3b^{-1} = a^{12}$; $ba^{12} = a^6b \implies ba^{12}b^{-1} = a^6$; $ba^6 = a^3b \implies ba^6b^{-1} = a^3$. The conjugate class corresponding is $\{a^3, a^{12}, a^6\}$.

5. Elements conjugate of a^5:

   $ba^5 = a^{20}b \implies ba^5b^{-1} = a^{20}$; $ba^{20} = a^{17}b \implies ba^{20}b^{-1} = a^{17}$; $ba^{17} = a^5b \implies ba^{17}b^{-1} = a^5$. The conjugate class corresponding is $\{a^5, a^{20}, a^{17}\}$.

6. Elements conjugate of a^7:

   $ba^7 = (a^7)^4b = a^{28}b = a^7b \implies ba^7b^{-1} = a^7$. The conjugate class corresponding is $\{a^7\}$.

7. Elements conjugate of a^9:

   $ba^9 = a^{15}b \implies ba^9b^{-1} = a^{15}$; $ba^{15} = a^{18}b \implies ba^{15}b^{-1} = a^{18}$; $ba^{18} = a^9b \implies ba^{18}b^{-1} = a^9$. The conjugate class corresponding is $\{a^9, a^{15}, a^{18}\}$.

8. Elements conjugate of a^{10}:

   $ba^{10} = a^{19}b \implies ba^{10}b^{-1} = a^{19}$; $ba^{19} = a^{13}b \implies ba^{19}b^{-1} = a^{13}$; $ba^{13} = a^{10}b \implies ba^{13}b^{-1} = a^{10}$. The conjugate class corresponding is $\{a^{10}, a^{19}, a^{13}\}$.

9. Elements conjugate of a^{14}:

   $ba^{14} = (a^{14})^4b = a^{56}b = a^{14}b \implies ba^{14}b^{-1} = a^{14}$. The conjugate class corresponding is $\{a^{14}\}$.

10. Elements conjugate of b:

    $ba = a^4b \implies a^{-1}ba = a^3b$; $ba = a^4b \implies a^3ba = a^7b \implies a^3b = a(a^6b)a^{-1}$; $a^6ba = a^{10}b \implies a^6b = a(a^9b)a^{-1}$; $a^9ba = a^{13}b \implies a^9b = a(a^{12}b)a^{-1}$; $a^{12}ba = a^{16}b \implies a^{12}b = a(a^{15}b)a^{-1}$; $a^{15}ba = a^{19}b \implies a^{15}b = a(a^{18}b)a^{-1}$; $a^{18}ba = a^{22}b = ab \implies a^{18}b = aba^1$. The conjugate class corresponding is $\{b, a^3b, a^6b, a^9b, a^{12}b, a^{15}b, a^{18}b\}$.

11. Elements conjugate of ab:

    $ba = a^4b \implies aba = a^5b \implies ab = a(a^4ba^{-1})$; $ba = a^4b \implies a^4ba = a^8b \implies a^4b = a(a^7b)a^{-1}$; $a^7ba = a^{11}b \implies a^7b = a(a^{10}b)a^{-1}$; $a^{10}ba = a^{14}b \implies a^{10}b = a(a^{13}b)a^{-1}$; $a^{13}ba = a^{17}b \implies a^{13}b = a(a^{16}b)a^{-1}$; $a^{16}ba = a^{20}b \implies a^{16}b = a(a^{19}b)a^{-1}$; $a^{19}ba = a^2b \implies a^{19}b = a(ab)a^1$. The conjugate class corresponding is $\{ab, a^4b, a^7b, a^{10}b, a^{13}b, a^{16}b, a^{19}b\}$.

12. Elements conjugate of a^2b:

    $ba = a^4b \implies a^2ba = a^6b \implies a^2b = a(a^5ba^{-1})$; $ba = a^4b \implies a^5ba = a^9b \implies a^5b = a(a^8b)a^{-1}$; $a^8ba = a^{12}b \implies a^8b = a(a^{11}b)a^{-1}$; $a^{11}ba = a^{15}b \implies a^{11}b = a(a^{14}b)a^{-1}$; $a^{14}ba = a^{18}b \implies a^{14}b = a(a^{17}b)a^{-1}$; $a^{17}ba = a^{21}b \implies a^{17}b = a(a^{20}b)a^{-1}$; $a^{20}ba = a^{24}b = a^3 \implies a^{20}b = a(a^2b)a^1$. The conjugate class corresponding is $\{a^2b, a^5b, a^8b, a^{11}b, a^{14}b, a^{17}b, a^{20}b\}$.

13. Elements conjugate of b^2:

    $ba = a^4b \implies b^2a = ba^4b = a^{16}b^2 \implies b^2 = a(a^{15}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{15}b^2a = a^{10}b^2 \implies a^{16}b^2 = a(a^9b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^9b^2a = a^4b^2 \implies a^{10}b^2 = a(a^3b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^3b^2a = a^{19}b^2 \implies a^3b^2 = a(a^{18}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{18}b^2a = a^{13}b^2 \implies a^{18}b^2 = a(a^{12}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{12}b^2a = a^7b^2 \implies a^{12}b^2 = a(a^6b^2)a^{-1}$;

    $b^2a = a^{16}b^2 \implies a^6b^2a = a^{22}b^2 \implies a^6b^2 = a(a^{21}b^2)a^{-1} = a(b^2)a^{-1}$.

    The conjugate class corresponding is
    $\{b^2, a^{15}b^2, a^9b^2, a^3b^2, a^{18}b^2, a^{12}b^2, a^6b^2\}$.

14. Elements conjugate of ab^2:

    $b^2a = a^{16}b \implies ab^2a = a^{17}b^2 \implies ab^2 = a(a^{16}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{16}b^2a = a^{11}b^2 \implies a^{16}b^2 = a(a^{10}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{10}b^2a = a^5b^2 \implies a^{10}b^2 = a(a^4b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^4b^2a = a^{20}b^2 \implies a^4b^2 = a(a^{19}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{19}b^2a = a^{14}b^2 \implies a^{19}b^2 = a(a^{13}b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^{13}b^2a = a^8b^2 \implies a^{13}b^2 = a(a^7b^2)a^{-1}$;
    $b^2a = a^{16}b^2 \implies a^7b^2a = a^2b^2 \implies a^7b^2 = a(ab^2)a^{-1}$.

The conjugate class corresponding is
$\{ ab^2, a^{16}b^2, a^{10}b^2, a^4b^2, a^{19}b^2, a^{13}b^2, a^7b^2\}$.

15. Elements conjugate of a^2b^2:
$b^2a = a^{16}b \implies a^2b^2a = a^{18}b^2 \implies a^2b^2 = a(a^{17}b^2)a^{-1}$; $b^2a = a^{16}b^2 \implies a^{17}b^2a = a^{12}b^2 \implies a^{17}b^2 = a(a^{11}b^2)a^{-1}$; $b^2a = a^{16}b^2 \implies a^{11}b^2a = a^6b^2 \implies a^{11}b^2 = a(a^5b^2)a^{-1}$; $b^2a = a^{16}b^2 \implies a^5b^2a = a^{21}b^2 \implies a^5b^2 = a(a^{20}b^2)a^{-1}$; $b^2a = a^{16}b^2 \implies a^{20}b^2a = a^{15}b^2 \implies a^{20}b^2 = a(a^{14}b^2)a^{-1}$; $b^2a = a^{16}b^2 \implies a^{14}b^2a = a^9b^2 \implies a^{14}b^2 = a(a^8b^2)a^{-1}$; $b^2a = a^{16}b^2 \implies a^8b^2a = a^3b^2 \implies a^8b^2 = a(a^2b^2)a^{-1}$.
The conjugate class corresponding is $\{a^2b^2, a^{17}b^2, a^{11}b^2, a^5b^2, a^{20}b^2, a^{14}b^2, a^8b^2\}$.

Therefore, the summary of the conjugacy classes for $G_{4,3}$ is presented in Table 3 as follows:

Number of class	Conjugacy class
1	$\{1\}$
2	$\{a, a^4, a^{16}\}$
3	$\{a^2, a^8, a^{11}\}$
4	$\{a^3, a^{12}, a^6\}$
5	$\{a^5, a^{20}, a^{17}\}$
6	$\{a^7\}$
7	$\{a^9, a^{15}, a^{18}\}$
8	$\{a^{10}, a^{19}, a^{13}\}$
9	$\{a^{14}\}$
10	$\{b, a^3b, a^6b, a^9b, a^{12}b, a^{15}b, a^{18}b\}$
11	$\{ab, a^4b, a^7b, a^{10}b, a^{13}b, a^{16}b, a^{19}b\}$
12	$\{a^2b, a^5b, a^8b, a^{11}b, a^{14}b, a^{17}b, a^{20}b\}$
13	$\{b^2, a^{15}b^2, a^9b^2, a^3b^2, a^{18}b^2, a^{12}b^2, a^6b^2\}$
14	$\{ab^2, a^{16}b^2, a^{10}b^2, a^4b^2, a^{19}b^2, a^{13}b^2, a^7b^2\}$
15	$\{a^2b^2, a^{17}b^2, a^{11}b^2, a^5b^2, a^{20}b^2, a^{14}b^2, a^8b^2\}$

Table 3. Conjugacy classes for $G_{4,3}$

3.2.2 Irreducible representations

We have the following conditions for a degree 1 representation ρ:
$\rho(a^{21}) = \rho(1) = 1$, $\rho(b^3) = \rho(a^7)$, and $\rho(ba) = \rho(a^4b)$.
From the first relation, $\rho(a)$ must be a 21st root of unity. Let $\zeta = e^{2\pi i/21}$, then $\rho(a) = \zeta^k$ for some integer k. From the second relation, $\rho(b)^3 = \rho(a^7) = \zeta^{7k}$. Finally, we have:

$$\rho(b)\rho(a) = \rho(a^4)\rho(b) \implies \rho(b)\zeta^k = \zeta^{4k}\rho(b).$$

Thus, $\rho(b)$ must satisfy $\rho(b)(\zeta^k - \zeta^{4k}) = 0$. We know that $\rho(b) \neq 0$, then $\zeta^k - \zeta^{4k} = 0 \implies \zeta^{3k} = 1$. This further implies that $3k \equiv 0 \mod 21$. It simplifies to $k \equiv 0 \mod 7$ Therefore, $k = 0, 7, 14$ are the possible values. For each of these:

- $\rho(a) = \zeta^0 = 1$ and $\rho(b) = \begin{cases} 1 \\ w \\ w^2 \end{cases}$ (where $\omega = e^{2\pi i/3}$).
- $\rho(a) = \zeta^7 = e^{2\pi i/3}$, and $\rho(b) = \begin{cases} e^{2\pi i/9} = \eta \\ e^{8\pi i/9} = \eta^4 \\ e^{14\pi i/9} = \eta^7 \end{cases}$
- $\rho(a) = \zeta^{14} = (e^{2\pi i/3})^2$, and $\rho(b) = \begin{cases} e^{4\pi i/9} = \eta^2 \\ e^{10\pi i/9} = \eta^5 \\ e^{16\pi i/9} = \eta^8 \end{cases}$

A 3-dimensional representation $\rho : G_{4,3} \to \mathrm{GL}(3, \mathbb{C})$ must satisfy the following conditions:
$\rho(a^{21}) = \rho(1) = I_3$, the 3×3 identity matrix, $\rho(b^3) = \rho(a^7)$, and $\rho(ba) = \rho(a^4b)$.
Let

$$\rho(a) = \begin{pmatrix} \zeta^r & 0 & 0 \\ 0 & \zeta^k & 0 \\ 0 & 0 & \zeta^l \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z & 0 & 0 \end{pmatrix}.$$

Here, z is a cube root of ζ^7, which ensures that $\rho(b^3) = \rho(a^7)$.

- $\rho(a) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$ and $\rho(b) = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta^4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta^7 & 0 & 0 \end{pmatrix} \right.$
- $\rho(a) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$ and $\rho(b) = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta^5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta^8 & 0 & 0 \end{pmatrix} \right.$

Here, we have $\omega = e^{2\pi i/3}$ and $\eta = e^{2\pi i/9}$.

3.2.3 Character table of $G_{4,3}$ by hand

We are going to fill in the character table of the Multiplicative Group $G_{4,3}$ by hand. This is presented in the Figure 1.

	1	a	a^2	a^3	a^5	a^7	a^9	a^{10}	a^{14}	b	ab	a^2b	b^2	ab^2	a^2b^2
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
χ_3	1	1	1	1	1	1	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
χ_4	1	ω	ω^2	ω^3	ω^5	ω^7	ω^9	ω^{10}	ω^{14}	η	$\omega \times \eta$	$\omega^2 \times \eta$	η^2	$\omega \times \eta^2$	$\omega^2 \times \eta^2$
χ_5	1	ω	ω^2	ω^3	ω^5	ω^7	ω^9	ω^{10}	ω^{14}	η^4	$\omega \times \eta^4$	$\omega^2 \times \eta^4$	η^8	$\omega \times \eta^8$	$\omega^2 \times \eta^8$
χ_6	1	ω	ω^2	ω^3	ω^5	ω^7	ω^9	ω^{10}	ω^{14}	η^7	$\omega \times \eta^7$	$\omega^2 \times \eta^7$	η^{14}	$\omega \times \eta^{14}$	$\omega^2 \times \eta^{14}$
χ_7	1	ω^2	ω^4	ω^6	ω^{10}	ω^{14}	ω^{18}	ω^{20}	ω^{28}	η^2	$\omega^2\eta^2$	$\omega^4\eta^2$	η^4	$\omega^2\eta^4$	$\omega^4\eta^4$
χ_8	1	ω^2	ω^4	ω^6	ω^{10}	ω^{14}	ω^{18}	ω^{20}	ω^{28}	η^5	$\omega^2 \times \eta^5$	$\omega^4 \times \eta^5$	η^{10}	$\omega^2 \times \eta^{10}$	$\omega^4 \times \eta^{10}$
χ_9	1	ω^2	ω^4	ω^6	ω^{10}	ω^{14}	ω^{18}	ω^{20}	ω^{28}	η^8	$\omega^2 \times \eta^8$	$\omega^4 \times \eta^8$	η^{16}	$\omega^2 \times \eta^{16}$	$\omega^4 \times \eta^{16}$
χ_{10}	3	3ω	$3\omega^2$	$3\omega^3$	$3\omega^5$	$3\omega^7$	$3\omega^9$	$3\omega^{10}$	$3\omega^{14}$	0	0	0	0	0	0
χ_{11}	3	3ω	$3\omega^2$	$3\omega^3$	$3\omega^5$	$3\omega^7$	$3\omega^9$	$3\omega^{10}$	$3\omega^{14}$	0	0	0	0	0	0
χ_{12}	3	3ω	$3\omega^2$	$3\omega^3$	$3\omega^5$	$3\omega^7$	$3\omega^9$	$3\omega^{10}$	$3\omega^{14}$	0	0	0	0	0	0
χ_{13}	3	$3\omega^2$	$3\omega^4$	$3\omega^6$	$3\omega^{10}$	$3\omega^{14}$	$3\omega^{18}$	$3\omega^{20}$	$3\omega^{28}$	0	0	0	0	0	0
χ_{14}	3	$3\omega^2$	$3\omega^4$	$3\omega^6$	$3\omega^{10}$	$3\omega^{14}$	$3\omega^{18}$	$3\omega^{20}$	$3\omega^{28}$	0	0	0	0	0	0
χ_{15}	3	$3\omega^2$	$3\omega^4$	$3\omega^6$	$3\omega^{10}$	$3\omega^{14}$	$3\omega^{18}$	$3\omega^{20}$	$3\omega^{28}$	0	0	0	0	0	0

Figure 1. Character table of $G_{4,3}$ by hand

3.2.4 Building character table of $G_{4,3}$ in Magma

We employ the Magma software to compute the character table of $G_{4,3}$.
Magma code:

```
q := 4; n := 3;
m := (q^n - 1) div (q - 1);
t := m div (q - 1);

F<a, b> := FreeGroup(2);
G := quo< F | a^m = 1, b^n = a^t, b*a = a^q*b >;

// Convert to a permutation group
G_4_3 := PermutationGroup(G);

// Compute the character table
CT := CharacterTable(G_4_3);

// Print the character table
CT;
```

Listing 2. Magma code for $G_{4,3}$

We receive the following result:

```
Character Table of Group G_4_3
------------------------------


-------------------------------------------------------------------------------
Class |  1    2    3    4    5    6    7    8    9   10   11   12   13
Size  |  1    1    1    3    3    7    7    7    7    7    7    3    3
Order |  1    3    3    7    7    9    9    9    9    9    9   21   21
-------------------------------------------------------------------------------
p = 3  1    1    1    5    4    2    3    2    3    2    3    4    4
p = 7  1    2    3    1    1   10    9    6   11    8    7    2    3
-------------------------------------------------------------------------------
X.1  +  1    1    1    1    1    1    1    1    1    1    1    1    1
X.2  0  1    1    1    1    1    J -1-J    J -1-J    J -1-J    1    1
X.3  0  1    1    1    1    1 -1-J    J -1-J    J -1-J    J    1    1
X.4  0  1 -1-J    J    1    1   Z2 Z2#2 Z2#4 Z2#5 Z2#7 Z2#8 -1-J    J
X.5  0  1 -1-J    J    1    1 Z2#7 Z2#5   Z2 Z2#8 Z2#4 Z2#2 -1-J    J
X.6  0  1    J -1-J    1    1 Z2#8 Z2#7 Z2#5 Z2#4 Z2#2   Z2    J -1-J
X.7  0  1    J -1-J    1    1 Z2#2 Z2#4 Z2#8   Z2 Z2#5 Z2#7    J -1-J
X.8  0  1 -1-J    J    1    1 Z2#4 Z2#8 Z2#7 Z2#2   Z2 Z2#5 -1-J    J
X.9  0  1    J -1-J    1    1 Z2#5   Z2 Z2#2 Z2#7 Z2#8 Z2#4    J -1-J
X.10 0  3    3    3   Z1 Z1#3    0    0    0    0    0    0 Z1#3 Z1#3
X.11 0  3    3    3 Z1#3   Z1    0    0    0    0    0    0   Z1   Z1
X.12 0  3   3*J -3-3*J Z1#3   Z1    0    0    0    0    0    0   Z3 Z3#2
X.13 0  3   3*J -3-3*J   Z1 Z1#3    0    0    0    0    0    0 Z3#10 Z3#5
X.14 0  3 -3-3*J   3*J   Z1 Z1#3    0    0    0    0    0    0 Z3#5 Z3#10
X.15 0  3 -3-3*J   3*J Z1#3   Z1    0    0    0    0    0    0 Z3#2   Z3
```

Figure 2. Character table of $G_{4,3}$ by hand

```
--------------------
p = 3      5     5
p = 7      3     2
--------------------
X.1  +     1     1
X.2  0     1     1
X.3  0     1     1
X.4  0     J  -1-J
X.5  0     J  -1-J
X.6  0  -1-J     J
X.7  0  -1-J     J
X.8  0     J  -1-J
X.9  0  -1-J     J
X.10 0    Z1    Z1
X.11 0  Z1#3  Z1#3
X.12 0  Z3#5 Z3#10
X.13 0  Z3#2    Z3
X.14 0    Z3  Z3#2
X.15 0 Z3#10  Z3#5
```

Figure 3. Character table of $G_{4,3}$ by hand

```
Explanation of Character Value Symbols
--------------------------------------

# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

J = RootOfUnity(3)

Z1     = (CyclotomicField(7: Sparse := true)) ! [ RationalField() | 0, 1, 1, 0,
1, 0 ]

Z2     = (CyclotomicField(9: Sparse := true)) ! [ RationalField() | 0, 0, 1, 0,
0, 0 ]

Z3     = (CyclotomicField(21: Sparse := true)) ! [ RationalField() | 0, 0, 0, 1,
0, 1, 0, 0, 0, 1, 0, 0 ]
```

Figure 4. Character table of $G_{4,3}$ by hand

3.3 Generalization for conjugacy classes for any pair (q, n)

Our goal is to find a generalization of the conjugacy classes of the multiplicative group $G_{q,n}$. Recall that:

$$G_{q,n} = \langle a, b \mid a^m = 1, b^n = a^t, ba = a^q b \rangle,$$

where $t = \frac{m}{q-1}$ and $m = \frac{q^n-1}{q-1}$.

The elements of $G_{q,n}$ are:

$$1, a, a^2, \ldots, a^{m-1}, b, ab, a^2b, \ldots, a^{m-1}b, b^2, ab^2, \ldots, a^{m-1}b^2,$$

$$\ldots, b^{n-1}, ab^{n-1}, \ldots, a^{m-1}b^{n-1}.$$

Now, we try to determine the shape of the conjugacy classes of the group $G_{q,n}$.

- Conjugacy classes of a^k: For a^k where $0 \leq k < m$, consider the action of b: $ba^kb^{-1} = (ba^k)b^{-1} = (a^{kq}b)b^{-1} = a^{kq}$. Thus, the elements a^k are permuted by powers of q. We need to find the distinct orbit under this action:

  $$\{a^k, a^{kq}, a^{kq^2}, \ldots, a^{kq^{r-1}}\},$$

  where r is the smallest integer such that $\text{kq}^r \equiv k \pmod{m}$.

Each such distinct set forms a single conjugacy class: $\{a^k, a^{kq}, a^{kq^2}, \ldots\}$.

- Conjugacy classes of a^kb^j: For a^kb^j where $0 \leq k < m$ and $1 \leq j < n$, consider the action of b:

  $$b(a^kb^j)b^{-1} = (ba^k)b^jb^{-1} = a^{kq}b^j.$$

Thus, elements a^kb^j are permuted by $a^{kq}b^j$. Additionally, we need to consider the action of a: $a(a^kb^j)a^{-1} = aa^kb^ja^{-1} = a^{k+1}b^j$. Therefore, a^kb^j conjugates to $a^{k+1}b^j$. The distinct elements form a single conjugacy class, considering both actions: $\{a^kb^j, a^{k+1}b^j, \ldots, a^{k+m-1}b^j\}$.

Now, we can state that:

Theorem 19. *Consider a Dickson pair (q, n) characterized by $q = p^l$, where p is a prime and l and n are positive integers. Let $G_{q,n}$ be the multiplicative group of any pair (q, n). The conjugacy classes of $G_{q,n}$ are given by:*

19. *Identity class :* $\{1\}$,

19. *Conjugacy classes of* a^k*:*

    For each k *such that* $0 \leq k < m$, $\{a^k, a^{kq}, a^{kq^2}, \ldots, a^{kq^{r-1}}\}$, *where* r *is the order of* q *modulo* m.

19. *Conjugacy classes of* $a^k b^j$*:*

    For each k *such that* $0 \leq k < m$ *and* $1 \leq j < n$, $\{a^k b^j, a^{k+1} b^j, \ldots, a^{k+m-1} b^j\}$.

Proof. We will do the proof by induction.

1. Base case: For $n = 1$, $G_{q,1}$ simplifies to: $G_{q,1} = \langle a, b \mid a^m = 1, b = a^t, ba = a^q b \rangle$.
   Since $b = a^t$, we have: $G_{q,1} = \langle a \mid a^m = 1 \rangle \cong \mathbb{Z}_m$.
   The conjugacy classes are: $\{1\}, \{a\}, \{a^2\}, \ldots, \{a^{m-1}\}$.
   These match the form $\{a^k\}$ for $k = 0, 1, \ldots, m-1$, confirming the base case.

2. Induction hypothesis: Assume the theorem holds for $n = k$. That is, the conjugacy classes for $G_{q,k}$ are as stated in the theorem.

3. Induction step: $n = k + 1$: Consider $G_{q,k+1}$: $G_{q,k+1} = \langle a, b \mid a^m = 1, b^{k+1} = a^t, ba = a^q b \rangle$. Examine the conjugacy classes of $G_{q,k+1}$ by considering both a^k and $a^k b^j$ for $0 \leq k < m$ and $1 \leq j \leq k$.

   - Identity class: $\{1\}$
   - Conjugacy classes of a^k: For a^k, the action of b permutes these elements as follows: $ba^k b^{-1} = a^{kq}$. By induction hypothesis, the conjugacy class of a^k is: $\{a^k, a^{kq}, a^{kq^2}, \ldots, a^{kq^{r-1}}\}$, where r is the order of q modulo m.
   - Conjugacy classes of $a^k b^j$: For $a^k b^j$ with $1 \leq j \leq k+1$, we consider two actions: conjugation by b and conjugation by a.
     - Conjugation by b: $b(a^k b^j)b^{-1} = a^{kq} b^j$.
     - Conjugation by a: $a(a^k b^j)a^{-1} = aa^k b^j a^{-1} = a^{k+1} b^j$.

   Combining these, the conjugacy class of $a^k b^j$ forms as: $\{a^k b^j, a^{k+1} b^j, \ldots, a^{k+m-1} b^j\}$. By induction, the theorem holds for all n. Thus, the conjugacy classes of $G_{q,n}$ are:

   - Identity class: $\{1\}$,
   - Conjugacy Classes of a^k: $\{a^k, a^{kq}, a^{kq^2}, \ldots, a^{kq^{r-1}}\}$,
   - Conjugacy classes of $a^k b^j$: $\{a^k b^j, a^{k+1} b^j, \ldots, a^{k+m-1} b^j\}$.

□

3.4 Generalization of the characters degree for any pair (q, n)

We provide a relationship between the character degree and the order of the multiplicative group $G_{q,n}$.

Theorem 20. *Let* $\deg \chi_i$ *be the degree of the character* χ*. We have:*

$$\sum_{i=1}^{k} (\deg \chi_i)^2 = |G_{q,n}|,$$

where k *is the number of irreducible characters of* $G_{q,n}$ *and* χ_i *is the* i*-th irreducible character.*

Proof. Let $\chi_1, \chi_2, \ldots, \chi_k$ be the irreducible characters of $G_{q,n}$.

The degree of each character χ_i, denoted $\deg \chi_i$, is the value of χ_i at the identity element of the group. Apply the orthogonality relations: $\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}|G|$, where δ_{ij} is the Kronecker delta, which is 1 if $i = j$ and 0 otherwise.
By setting $i = j$ and summing over all elements g in the group, we have: $\sum_{g \in G} |\chi_i(g)|^2 = |G|$. When we sum this equation over all irreducible characters, we get: $\sum_{i=1}^{k} \sum_{g \in G} |\chi_i(g)|^2 = \sum_{i=1}^{k} |G|$. Simplifying, and knowing that the left-hand sum counts each group element once for each character degree squared, gives: $\sum_{i=1}^{k} (\deg \chi_i)^2 = |G|$.
□

3.5 Generalization of irreducible representations for any pair (q, n)

Now we will give a generalization of the irreducible representations of the multiplicative group $G_{q,n}$ based on the representation of the group.

Theorem 21. *Let* a *and* b *belong to the multiplicative group* $G_{q,n}$ *with* $a^m = 1, b^n = a^t, ba = a^q b$*. For any Dickson pair* (q, n) *characterized by* $q = p^l$*, where* p *is a prime and* l *and* n *are positive integers., the irreducible representations of* a *and* b *are given by:*

1. *for* $\rho(a)$ *:*

$$\rho(a) = \begin{bmatrix} \omega^{k_1} & 0 & \cdots & 0 \\ 0 & \omega^{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{k_r} \end{bmatrix},$$

   where $\omega = e^{2\pi i/m}$ *be a primitive* m*-th root of unity, and consider distinct integers* $k_1, k_2, \ldots, k_r$ *modulo* m.

2. *Pour* $\rho(b)$ *:*

$$\rho(b) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_r & 0 & 0 & \cdots & 0 \end{bmatrix},$$

   where c_r *is an* n*-th root of* ω^t*, ensuring* $\rho(b)^n = \rho(a^t)$.

Proof. we need to verify that these matrices satisfy the given relations of $G_{q,n}$.

1 Verify $\rho(a^m) = I$
Since ω is a primitive m-th root of unity, we have:

$\omega^m = 1$.
Thus, for $\rho(a)$:

$$\rho(a)^m = \begin{bmatrix} (\omega^{k_1})^m & 0 & \cdots & 0 \\ 0 & (\omega^{k_2})^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\omega^{k_r})^m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I.$$

So, $\rho(a^m) = I$, which satisfies the relation $a^m = 1$.

2 Verify $\rho(b^n) = \rho(a^t)$
We compute $\rho(b^n)$ using the matrix form of $\rho(b)$:

$$\rho(b) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_r & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

To find $\rho(b^n)$, we multiply $\rho(b)$ by itself n times. This results in:

$$\rho(b)^n = \begin{bmatrix} (c_r) & 0 & 0 & \cdots & 0 \\ 0 & c_r & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & c_r \end{bmatrix} = c_r I.$$

We know that c_r must be an n-th root of ω^t such that $c_r^n = \omega^t$. Therefore:

$\rho(b)^n = \rho(a^t)$ as required by the relation $b^n = a^t$.

3 Verify $\rho(ba) = \rho(a^q b)$

To verify $\rho(ba) = \rho(a^q b)$, we compute: $\rho(a)\rho(b) = \begin{bmatrix} \omega^{k_1} & 0 & \cdots & 0 \\ 0 & \omega^{k_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \omega^{k_r} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_r & 0 & 0 & \cdots & 0 \end{bmatrix}$.

This gives: $\rho(a)\rho(b) = \begin{bmatrix} 0 & \omega^{k_1} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{k_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{k_{r-1}} \\ c_r\omega^{k_r} & 0 & 0 & \cdots & 0 \end{bmatrix}$.

On the other hand, we have:

$$\rho(a^q)\rho(b) = \begin{bmatrix} \omega^{qk_1} & 0 & \cdots & 0 \\ 0 & \omega^{qk_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{qk_r} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_r & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This gives:

$$\rho(a^q)\rho(b) = \begin{bmatrix} 0 & \omega^{qk_1} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{qk_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{qk_{r-1}} \\ c_r\omega^{qk_r} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For $\rho(a^q b)$ to equal $\rho(a)\rho(b)$, we require $qk_i \equiv k_i \mod m$. Hence, $\rho(ba) = \rho(a^q b)$.

Thus, we have verified that the given forms for $\rho(a)$ and $\rho(b)$ satisfy all the defining relations of the group $G_{q,n}$. Hence, the theorem is proved.

□

3.6 Magma code for the character table of the Multiplicative Group $G_{q,n}$

We will provide a Magma code that generates the character table of any multiplicative group $G_{q,n}$.

Magma code:

```
q := \\ value of q; n := \\ value of n;
m := (q^n - 1) div (q - 1);
t := m div (q - 1);

F<a, b> := FreeGroup(2);
G := quo< F | a^m = 1, b^n = a^t, b*a = a^q*b >;

// Convert to a permutation group
G_q_n := PermutationGroup(G);

// Compute the character table
CT := CharacterTable(G_q_n);

// Print the character table
CT;
```

Listing 3. Magma code for any $G_{q,n}$

4 Conclusions

This study deepens our understanding of the multiplicative groups of finite near-field and their associated representations. The results obtained provide a solid foundation for future research in the generalization of representation theories of properties about the multiplicative groups of finite near-fields.

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